Chapter 3.1 part 1

Chapter 3.1 Definition of Ring

Recall Both $\mathbb{Z}$ and $\mathbb{Z}_n$ are sets equipped with operations - addition and multiplication

Operation is a rule to produce an element of the set out of an ordered pair of its elements: $a+b$ $[a]\odot[b]$ ...

Examples: $\mathbb{Z}$, $\mathbb{Z}_n$, $\mathbb{Q}$, $\mathbb{R}$, $M(\mathbb{R})$, $f: \mathbb{R} \to \mathbb{R}$

In many cases, elements of the set cannot be taken "numbers".

Def A ring is a non-empty set R equipped with two operations - addition and multiplication which satisfy the following axioms

a, b, c - arbitrary elements of R

(1) $a \in R$, $b \in R$ implies $a+b \in R$

(6) $a \in R$, $b \in R$ implies $ab \in R$

(2) $a+(b+c) = (a+b)+c$

(7) $a(bc) = (ab)c$

associativity of addition & multipli.

(8) $a(b+c) = ab+ac$
$(a+b)c = ac+bc$

Distributive laws

Extra requirements for the addition

(3) $a+b=b+a$ addition is commutative

(4) There is $0_R \in R$ which satisfies

$a+0_R=a \quad (=0_R+a)$

(5) For each $a\in R$, the equation

$a+x=0_R$ has a solution in R

"element $-a$"

Examples

$E=\{x\in\mathbb{Z} \mid x \text{ is even}\}$ is a ring (Ex 3)

In contrast, $\{x\in\mathbb{Z} \mid x \text{ is odd}\}$ is <u>not</u> a ring

(Ex 8) Functions $f:\mathbb{R}\to\mathbb{R}$ is a ring

$(f+g)(x)=f(x)+g(x)$

$(fg)(x)=f(x)g(x)$

In contrast, functions $f:\mathbb{R}\to\mathbb{R}$ with operations

$(f+g)(x)=f(x)+g(x)$

$(fg)(x) = f(g(x))$ is <u>not</u> a ring: $f(g+h)(x) = f(g(x)+h(x))$

$$\neq f(g(x)) + f(h(x))$$

- distributive law fails